J. Appl. Maths Mechs Vol. 57, No. 3, pp. 537–545, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

THE STABILITY OF MOTION OF A STOCHASTIC VISCOELASTIC SYSTEM†

V. D. Potapov

Moscow

(*Received* 8 *July* 1992)

The sufficient conditions for stability are obtained for the null solution of a system of integro-differential equations under rather general assumptions on the kernels of the integral operators and the form of the stochastic coefficients. Similar conditions of stability are also found for the case of an exponential kernel and stochastic coefficients in the form of Gaussian white noise in order to find a broader stability domain in the system parameter space. The root-mean-square stability of the null solution of the equation of motion of a viscoelastic rod has already been analysed in earlier publications; here the stability of a viscoelastic system is considered in probabilistic terms.

Considerable difficulties arise in solving the problem of the stability of systems of integro-differential equations with random parameters, including, in particular, those that describe the behaviour of viscoelastic systems under the action of random loads. If the viscosity of the construction material is low, and the root-mean-square random load fluctuations are also small, these difficulties can be overcome using the asymptotic method [1, 2]. For an integral operator kernel that can be represented in the form of the sum of exponential functions, and loads that are Gaussian white noise, necessary and sufficient conditions were found in [3, 4] for stability with respect to statistical moments. Using the same assumptions about the load and kernel, sufficient conditions for almost sure stability were found in [5] which are identical with the root-mean-square stability conditions [3, 4] when there is no external damping. Sufficient conditions have been found [6] for the root-mean-square stability for the general case of a material creep kernel and a longitudinal force in the form of a delta-correlated stationary process.

1. THE GENERAL CASE

Below the solution of a system of equations will be called almost surely stable [7–9] (strongly probabilistically stable [10]) for $t>0$ if

$$P\left\{\lim_{\|\mathbf{x}_0\|\to 0}\sup_{t\geq 0}\|\mathbf{x}(t,\mathbf{x}_0)\|=0\right\}=1$$

This condition can be represented in a somewhat different equivalent form

$$\lim_{\delta\to 0}P\left\{\sup_{\|\mathbf{x}_0\|<\delta}\sup_{t\geq 0}\|\mathbf{x}(t,\mathbf{x}_0)\|>\varepsilon\right\}=0$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 3, pp. 137–145, 1993.

where ε is an arbitrary small positive number.

By $\|\mathbf{x}(t)\|$, $\|\mathbf{x}_0\|$ we mean the norm of the solution at times t and $t=0$.

The solution $\mathbf{x}(t)$ is almost surely asymptotically stable if the preceding condition is satisfied, and furthermore, if there is a $\delta>0$ such that for $\|\mathbf{x}_0\|<\delta$ and any small $\varepsilon>0$

$$\lim_{t_1\to\infty} P\left\{\sup_{t>t_1}\|\mathbf{x}(t,\mathbf{x}_0)\|>\varepsilon\right\}=0$$

Consider the system of linear integrodifferential equations

$$\ddot{\mathbf{x}}=\mathbf{A}_1\mathbf{x}+\mathbf{G}_1\Gamma\mathbf{x}+\mathbf{D}_1\dot{\mathbf{x}} \tag{1.1}$$

$$\Gamma x_j=\int_0^t \hat{\Gamma}(t-\tau)x_j(\tau)d\tau$$

where $\mathbf{x}=\{x_j\}$ is the vector of unknown quantities, $j=1, 2,\ldots, n_1$, $\mathbf{A}_1$, $\mathbf{G}_1$ and $\mathbf{D}_1$ are square $(n_1\times n_1)$ matrices, and the kernel $\hat{\Gamma}(t-\tau)$ is a strictly monotonically decreasing function which satisfies the condition

$$0\leqslant\int_0^\infty\hat{\Gamma}(\theta)d\theta<1,\quad 0\leqslant\hat{\Gamma}(\theta)\text{ when }\theta\geqslant 0$$

By expanding the phase space one can write Eq. (1.1) in the form of a system of equations for the first derivatives

$$\dot{\mathbf{z}}=\mathbf{A}\mathbf{z}+\mathbf{G}\Gamma\mathbf{z} \tag{1.2}$$

The vector $\mathbf{z}$ contains $n=2n_1$ components, and $\mathbf{A}$ and $\mathbf{G}$ are square $(n\times n)$ matrices.

Below we shall assume that the matrix $\mathbf{A}$ can be written as a sum $\mathbf{A}=\mathbf{D}+\mathbf{F}(t)$ where the matrices $\mathbf{D}$ and $\mathbf{G}$ are constant (the matrix $\mathbf{D}$ is stable), while $\mathbf{F}(t)$ is a matrix whose elements are stochastic stationary ergodic processes.

The matrix $\mathbf{F}(t)$ can be represented as follows:

$$\mathbf{F}(t)=\sum_{k=1}^m f_k(t)\mathbf{F}_k,\quad m\leqslant n_1^2$$

where $\mathbf{F}(t)$ are constant matrices and $f_k(t)$ are random stationary functions that are assumed to be bounded, integrable and ergodic.

We will introduce the positive-definite quadratic form $V=\mathbf{z}^*\mathbf{P}\mathbf{z}$. Here and below an asterisk denotes transposition.

Using the notation $\mathbf{y}=\mathbf{P}^{1/2}\mathbf{z}$, the quadratic form V can be expressed as the square of the norm of the vector $\mathbf{y}$

$$V=\mathbf{y}^*\mathbf{y}=\|\mathbf{y}\|^2$$

The matrix $\mathbf{P}^{1/2}$ is given by the expression

$$\mathbf{P}^{1/2}=\mathbf{W}\mathbf{L}^{1/2}\mathbf{W}^*,\quad \mathbf{L}^{1/2}=\mathrm{diag}[\lambda_1^{1/2},\ldots,\lambda_n^{1/2}]$$

where $\mathbf{W}$ is the matrix of orthonormal eigenvectors of the symmetric matrix $\mathbf{P}$, and $\mathbf{L}$ is the diagonal matrix of eigenvalues of that matrix.

The derivative of the quadratic form along the trajectory of the solution of Eq. (1.2) is

$$\dot{V}=\mathbf{y}^*(\mathbf{B}^*+\mathbf{B})\mathbf{y}+\sum_{k=1}^m f_k(t)\mathbf{y}^*(\mathbf{C}_k^*+\mathbf{C}_k)\mathbf{y}+ \tag{1.3}$$
$$+\mathbf{y}^*\mathbf{H}(\Gamma\mathbf{y})+(\Gamma\mathbf{y})^*\mathbf{H}^*\mathbf{y}$$

$$\mathbf{B} = \mathbf{P}^{1/2}\mathbf{D}\mathbf{P}^{-1/2}, \quad \mathbf{C}_k = \mathbf{P}^{1/2}\mathbf{F}_k\mathbf{P}^{-1/2}, \quad \mathbf{H} = \mathbf{P}^{1/2}\mathbf{G}\mathbf{P}^{-1/2}$$

For quadratic forms we have the well-known inequalities [11]

$$\mathbf{y}^*(\mathbf{B}^* + \mathbf{B})\mathbf{y} \leq \eta\|y\|^2, \quad y^*(\mathbf{C}_k^* + \mathbf{C}_k)y \leq \mu_k\|y\|^2$$

Here η and μ_k are the largest eigenvalues of the matrices $\mathbf{B}^* + \mathbf{B}$ and $\mathbf{C}_k^* + \mathbf{C}_k$ (μ_k being the largest in absolute value).

Using the Cauchy–Bunyakovskii–Schwartz inequality [12], we have

$$\mathbf{y}^*\mathbf{H}(\Gamma\mathbf{y}) \leq n\rho\|\mathbf{y}(t)\|\int_0^t \hat{\Gamma}(t-\tau)\|\mathbf{y}(\tau)\|d\tau$$

$$\rho = \max_{i,j}|h_{ij}|$$

As a result, from (1.3) we obtain

$$2\frac{d}{dt}\|\mathbf{y}(t)\| \leq \left(\eta + \sum_{k=1}^m \mu_k|f_k(t)|\right)\|\mathbf{y}(t)\| + \tag{1.4}$$

$$+2n\rho\int_0^t \hat{\Gamma}(t-\tau)\|y(\tau)\|d\tau$$

We use the substitution

$$\|y(t)\| = e^{\eta t/2}r(t), \quad r(t) \geq 0$$

after which inequality (1.4) takes the form

$$\dot{r}(t) \leq \frac{1}{2}\sum_{k=1}^m \mu_k|f_k(t)|r(t) + n\rho\int_0^t \hat{\Gamma}(t-\tau)e^{-\eta(t-\tau)/2}r(\tau)d\tau$$

or

$$\dot{r}(t) \leq \frac{1}{2}\sum_{k=1}^m \mu_k|f_k(t)|\upsilon(t) + n\rho\upsilon(t)\int_0^t \hat{\Gamma}(\theta)e^{-\eta\theta/2}d\theta \tag{1.5}$$

$$\left(\upsilon(t) = \max_{\tau\in[0,t]} r(\tau)\right)$$

We will now assume that the kernel $\hat{\Gamma}(\theta)$ is such that the function

$$\Phi(t) = \int_0^t \hat{\Gamma}(\theta)e^{-\eta\theta/2}d\theta \geq 0 \tag{1.6}$$

is integrable and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(\tau)d\tau = c \quad c = \text{const}$$

Integrating both sides of inequality (1.5), we find that

$$r(t) \leq \upsilon(t) \leq \|\mathbf{y}(0)\| +$$

$$+\int_0^t\left[\frac{1}{2}\sum_{k=1}^m \mu_k|f_k(\tau)| + n\rho\Phi(\tau)\right]\upsilon(\tau)d\tau$$

Using the Gronwall–Bellman lemma [12] we have

$$\upsilon(t) \le \|y(0)\| \exp\left\{\int_0^t \left[\frac{1}{2}\sum_{k=1}^{m} \mu_k |f_k(\tau)| + n\rho\Phi(\tau)\right] d\tau\right\}$$

or

$$\|y(t)\| \le \|y(0)\| \exp\left\{\left[\frac{\eta}{2} + \frac{1}{2}\sum_{k=1}^{m}\mu_k \frac{1}{t}\int_0^t |f_k(\tau)| d\tau + n\rho\frac{1}{t}\int_0^t \Phi(\tau) d\tau\right] t\right\}$$

It is obvious here that as $t \to \infty$ the norm $\|y\|$ tends to zero if the expression in square brackets is a negative quantity.

From the ergodicity condition on the stationary functions $f_k(t)$ it follows that

$$\langle |f_k| \rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t |f_k(\tau)| d\tau$$

Here the angular brackets denote averaging over the statistical ensemble.

As a result the null solution of the system of integrodifferential equations (1.1) is almost surely asymptotically stable if the condition

$$\eta + \sum_{k=1}^{m} \mu_k \langle |f_k| \rangle + 2n\rho c < 0 \tag{1.7}$$

is satisfied.

Thus the ergodicity requirements on the stochastic coefficients of the equations enables one to formulate stability conditions for a broad class of stationary processes that can be used to describe parametric excitation of real viscoelastic systems.

Example. In the investigation of the stability of a compressed viscoelastic rod, hinged at both ends, the equation for the amplitude of a sinusoidal deflection (with initial perturbations also specified in the form of a sinusoid) has the form

$$\ddot{x} + 2\varepsilon\dot{x} + (1-\alpha)x - f(t)x - \Gamma x = 0 \tag{1.8}$$

where 2ε is a coefficient characterizing the effect of external damping, α is a dimensionless parameter for the constant external compressive force $(\alpha < 1)$, and $f(t)$ is a random stationary process proportional to the variable component of the longitudinal force with a mathematical expectation of zero.

The matrices $\mathbf{D}, \mathbf{G}$ and $\mathbf{F}$ have the form

$$\mathbf{D} = \begin{Vmatrix} 0 & 1 \\ -1+\alpha & -2\varepsilon \end{Vmatrix}, \quad \mathbf{G} = \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}, \quad \mathbf{F} = f(t)\mathbf{G}$$

We will take a kernel $\hat{\Gamma}(t-\tau)$ of the form

$$\hat{\Gamma}(t-\tau) = \kappa R \exp[-\kappa(t-\tau)] \quad (0 \le R < 1)$$

(κ and R being constants).

In this case

$$\Phi(t) = \kappa R(\kappa + \eta/2)^{-1}\{1 - \exp[-(\kappa+\eta/2)t]\}$$

$$I(t) = \frac{1}{t}\int_0^t \Phi(\tau)d\tau = \frac{\kappa R}{\kappa+\eta/2}\left\{1 - \frac{1}{(\kappa+\eta/2)t}\left[1 - e^{-(\kappa+\eta/2)t}\right]\right\}$$

If κ is such that $\kappa + \eta/2 > 0$, then

$$\lim_{t\to\infty} I(t) = c = \kappa R(\kappa + \eta/2)^{-1}$$

Then inequality (1.7) can be written as follows:

$$\eta + \mu\langle |f| \rangle + 4\rho c < 0$$

If $k \gg \eta/2$, one can put

$$\eta + \mu\langle |f| \rangle + 4\rho R < 0$$

From this we find

$$\langle |f| \rangle < -(\eta + 4\rho R)/\mu \tag{1.9}$$

We take as the matrix **P**

$$\mathbf{P} = \begin{Vmatrix} 1-\alpha+2\varepsilon^2 & \varepsilon \\ \varepsilon & 1 \end{Vmatrix}$$

which was obtained in [9] for Eq. (1.8) (when $\Gamma x \equiv 0$) from the condition for determining the largest domain of stability in the parameter space of the equation.

The eigenvalues and corresponding eigenvectors for this matrix are

$$\lambda_1 = 1-\alpha/2+\varepsilon^2+k,\quad \lambda_2 = 1-\alpha/2+\varepsilon^2-k$$

$$\mathbf{W}_1 = (1+r^2)^{-1/2}\begin{Vmatrix} 1 \\ -r \end{Vmatrix},\quad \mathbf{W}_2 = (1+r^2)^{-1/2}\begin{Vmatrix} r \\ 1 \end{Vmatrix}$$

$$k = [\varepsilon^4 + (1-\alpha)\varepsilon^2 + \alpha^2/4]^{1/2},\quad r = -\alpha/(2\varepsilon) + \varepsilon - k/\varepsilon$$

Figures 1–3 show graphs of the parameters η, μ and ρ as a function of α for different values of ε. Using these graphs with specified values of R and ε it is easy to estimate the quantity $\langle |f| \rangle$. We note that for small values of ε the quantity R is allowed to be small.

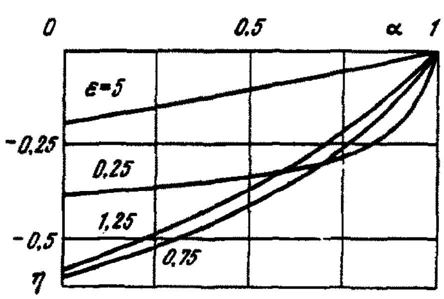

FIG. 1.

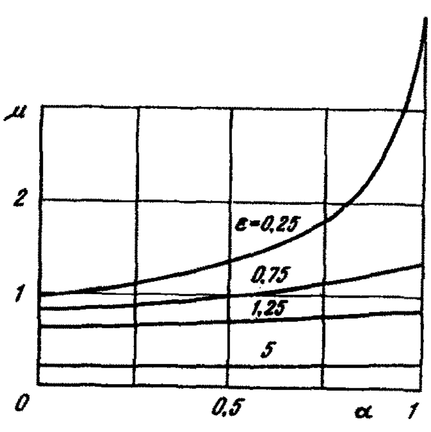

FIG. 2.

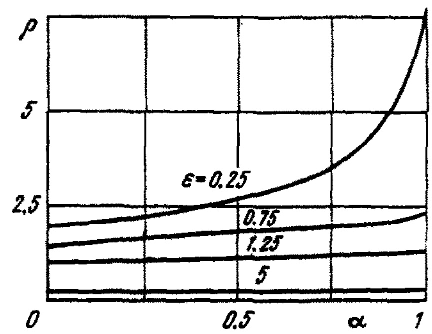

FIG. 3.

2. A SPECIAL CASE

The stability condition in the form (1.7) was obtained under rather broad assumptions about the form of the kernel of the integral operator Γx and stochastic processes $f_k(t)$. Because of this the estimates of characteristic parameters of the system ensuring almost sure stability of the solution of Eqs (1.1) are very restrictive, which in particular explains the rather rough estimates of the integral terms in inequality (1.4), and also the restriction on the nature of the variation of the kernel $\hat{\Gamma}(t-\tau)$ represented in the form (1.6). These restrictions can be weakened if we consider special cases of the kernel $\hat{\Gamma}(t-\tau)$ and functions $f_k(t)$.

In the theory of viscoelasticity it is often assumed that the kernel of the relaxation and material creep operators can be represented in the form of a sum of exponential functions

$$\hat{\Gamma}(t-\tau)=\sum_{j=1}^{l}\kappa_j R_j \exp[-\kappa_j(t-\tau)]$$

(κ_j and R_j are constants).

Below we consider a kernel of this type.

Using the substitution

$$\mathbf{u}_j=\int_0^t \kappa_j R_j \exp[-\kappa_j(t-\tau)]\mathbf{z}(\tau)d\tau$$

the system of integrodifferential equations (1.2) changes to a system of differential equations

$$\begin{aligned}&\dot{\mathbf{z}}=\mathbf{A}\mathbf{z}+\mathbf{G}\sum_{j=1}^{l}\mathbf{u}_j\\&\dot{\mathbf{u}}_j=\kappa_j R_j \mathbf{z}-\kappa_j\mathbf{u}_j\quad (j=1,2,\ldots,l)\end{aligned}\tag{2.1}$$

Equation (2.1) can again be written in the form

$$\dot{\mathbf{w}}=\left[\mathbf{S}+\sum_{k=1}^{m}f_k(t)\mathbf{M}_k\right]\mathbf{w}$$

$$\mathbf{S}=\begin{Vmatrix}\mathbf{A}&\mathbf{G}&\mathbf{G}&\ldots&\mathbf{G}\\\mathbf{E}_1&-\mathbf{N}_1&0&\ldots&0\\\mathbf{E}_2&0&-\mathbf{N}_2&\ldots&0\\\cdot&\cdot&\cdot&&\cdot\\\mathbf{E}_l&0&0&\ldots&-\mathbf{N}_l\end{Vmatrix},\quad \mathbf{M}_k=\begin{Vmatrix}\mathbf{F}_k&0&\ldots&0\\0&0&\ldots&0\\\cdot&\cdot&&\cdot\\\cdot&\cdot&&\cdot\\\cdot&\cdot&&\cdot\\0&0&\ldots&0\end{Vmatrix},\quad \mathbf{w}=\begin{Vmatrix}\mathbf{z}\\\mathbf{u}_1\\\cdot\\\cdot\\\cdot\\\mathbf{u}_l\end{Vmatrix}$$

$$\mathbf{E}_i = \kappa_i R_i \mathbf{E}, \quad \mathbf{N}_i = \kappa_i \mathbf{E}$$

(where $\mathbf{E}$ is the identity matrix).

Then from the matrix $\mathbf{S}$ one can construct a positive-definite quadratic form [13], and using it, almost literally repeating the argument of the previous section, obtain a stability condition in the form of an inequality similar to (1.7)

$$\eta + \sum_{k=1}^{m} \mu_k \langle |f_k| \rangle < 0$$

3. GAUSSIAN WHITE NOISE

If the stationary processes are Gaussian white noise, the above approach cannot be directly applied. In this case one has to use the theory of Markov processes [10]. Features of the solution of the problem using this approach will be considered using, as an example, the equation

$$\ddot{x} + (1-\alpha)x - \int_0^t \kappa R e^{-\kappa(t-\tau)} x(\tau) d\tau + 2\varepsilon\dot{x} - \beta\xi(t)x = 0 \tag{3.1}$$

where $\xi(t)$ is Gaussian white noise, $\varepsilon \geq 0$, $\kappa \geq 0$, $0 \leq R < 1$.

We will write Eq. (3.1) in the form of a system of first-order differential equations

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \xi \mathbf{C}\mathbf{x} \tag{3.2}$$

$$\mathbf{x} = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix}, \quad \mathbf{B} = \begin{Vmatrix} 0 & 1 & 0 \\ -1+\alpha & -2\varepsilon & 1 \\ \kappa R & 0 & -\kappa \end{Vmatrix}, \quad \mathbf{C} = \begin{Vmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

$$x_3 = \int_0^t \kappa R e^{-\kappa(t-\tau)} x_1(\tau) d\tau$$

We write out the positive-definite quadratic form

$$V = a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + x_3^2$$

(the a_{ij} are constants).

Its positive definiteness is ensured by the inequalities

$$a_{11} > 0, \quad a_{22} > 0, \quad a_{11} + a_{22} + a_{11}a_{22} - a_{12}^2 - a_{13}^2 - a_{23}^2 > 0,$$

$$a_{12}^2 + a_{23}^2 a_{11} + a_{22}a_{13}^2 - 2a_{12}a_{23}a_{13} - a_{11}a_{22} > 0$$

We know [10] that the null solution of system (3.2) is probabilistically stable in the strong sense if the condition

$$LV = \frac{\partial V}{\partial t} + \sum_{i=1}^{3} \sum_{j=1}^{3} b_{ij} x_j \frac{\partial V}{\partial x_i} + \frac{\beta^2}{2} x_1^2 \frac{\partial^2 V}{\partial x_2^2} \leq 0 \tag{3.3}$$

is satisfied.

After making all the substitutions we obtain

$$\tfrac{1}{2}LV = [-(1-\alpha)a_{12} + \kappa R a_{13} + \tfrac{1}{2}\beta^2 a_{22}]x_1^2 +$$
$$+[a_{11} - 2\varepsilon a_{12} - (1-\alpha)a_{22} + \kappa a_{23}R]x_1 x_2 +$$
$$+[-(1-\alpha)a_{23} + a_{12} + \kappa R - \kappa a_{13}]x_1 x_3 +$$
$$+(a_{12} - 2\varepsilon a_{22})x_2^2 + (a_{13} - 2\varepsilon a_{23} - \kappa a_{23} +$$
$$+a_{22})x_2 x_3 + (a_{23} - \kappa)x_3^2 \leqslant 0$$

We require the coefficient in front of x_1^2 to be negative, and all the others to be zero. Then

$$a_{11} = a(1-\alpha+4\varepsilon^2)\Delta - \kappa^2 R, \quad a_{12} = 2\varepsilon a\Delta$$
$$a_{22} = a\Delta, \quad a_{23} = \kappa, \quad a_{13} = a[(2\varepsilon+\kappa)^2 - \Delta]$$
$$\beta^2 < 2[2\varepsilon(1-\alpha) + \kappa R - \kappa R(2\varepsilon+\kappa)^2/\Delta] \tag{3.4}$$
$$(a = \kappa/(2\varepsilon+\kappa), \quad \Delta = 1 - R - \alpha + \kappa(2\varepsilon+\kappa))$$

Condition (3.3) reduces to the following inequalities

$$\Delta > 0, \quad (1-\alpha+4\varepsilon^2)\Delta - \kappa(2\varepsilon+\kappa)R > 0 \tag{3.5}$$
$$\Delta(2-\alpha+4\varepsilon^2-\kappa^2 R) - \alpha a\Delta^2 -$$
$$-\kappa(2\varepsilon+\kappa)[1+R-2\Delta+(2\varepsilon+\kappa)^2] > 0$$
$$(1-\alpha-R)\left\{[2\varepsilon(1-\alpha)+\kappa R]\Delta - \kappa(2\varepsilon+\kappa)^2 R\right\} > 0$$

Thus the null solution of Eq. (3.1) is almost surely stable if conditions (3.4) and (3.5) are satisfied.

It can be shown that for sufficiently small κ and $1-\alpha-R>0$ (the stability condition for the quasi-static formulation of the problem with $\beta=0$) inequality (3.5) is satisfied. Consequently, in this case the condition for stability is the satisfaction of inequality (3.4).

If $\kappa = R = 0$, then from (3.4) we have

$$\beta^2 < 4\varepsilon(1-\alpha) \tag{3.6}$$

If $\varepsilon = 0$ then

$$\beta^2 < 2\kappa R(1-\alpha-R)(1-\alpha-R+\kappa^2)^{-1}$$

We note that these estimates for β^2 are identical with similar estimates obtained from the root-mean-square condition for the solution to be stable.

Thus in this case the range of variation of β^2 is identical with that for the root-mean-square condition of stability of the solution $x \equiv 0$.

REFERENCES

1. POTAPOV V. D., *Stability of Viscoelastic Structural Elements.* Stroiizdat, Moscow, 1985.
2. DIMENTBERG M. F., *Random Processes in Dynamical Systems with Variable Parameters.* Nauka, Moscow, 1989.
3. POTAPOV V. D., Stability of a viscoelastic rod under the action of a random stationary longitudinal force. *Prikl. Mat. Mekh.* **56**, 1, 105–110, 1992.
4. POTAPOV V. D., On the stability of viscoelastic beams under a stochastic excitation. In *Creep in Structures*, IUTAM Symposium, Cracow, Poland, 1990, pp. 609–614. Springer, Berlin, 1991.

5. TYLIKOWSKII A., Stability and bounds on motion of viscoelastic columns with imperfections and time-dependent forces. In *Creep in Structures*, IUTAM Symposium, Cracow, Poland, 1990, pp. 653–658. Springer, Berlin, 1991.
6. DROZDOV A. D. and KOLMANOVSKII V. B., Stability of viscoelastic rods under a random longitudinal load. *Zh. Prikl. Mekh. Tekh. Fiz.* 5, 124–131, 1991.
7. CAUGHEY T. K. and GRAY A. H., On the almost sure stability of linear dynamical systems with stochastic coefficients. *Trans. ASME Ser. E, J. Appl. Mech.* **32**, 2, 365–372, 1965.
8. INFANTE E. F., On the stability of some linear non-autonomous random systems. *Trans. ASME Ser. E, J. Appl. Mech.* **35**, 1, 7–12, 1968.
9. KOZIN F. and WU C.-M., On the stability of linear stochastic differential equations. *Trans. ASME Ser. E, J. Appl. Mech.* **40**, 1, 87–92, 1973.
10. KHAS'MINSKII R. Z., *Stability of a System of Differential Equations with Random Perturbations of their Parameters.* Nauka, Moscow, 1969.
11. GANTMAKHER F. R., *Matrix Theory.* Nauka, Moscow, 1967.
12. BEKKENBACH E. and BELLMAN R., *Inequalities.* Mir, Moscow, 1965.
13. BARBASHIN Ye. A., *Lyapunov Functions.* Nauka, Moscow, 1970.

Translated by R.L.Z.